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Pfaffian solution of a semi-discrete BKP-type equation and its source generation version

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Abstract
The pfaffian solution of a semi-discrete BKP-type equation is obtained at first, then utilizing the source generation procedure, this equation with self-consistent sources (BKPESCS) is presented and its pfaffian solutions are derived. Finally, a bilinear Bäcklund transformation for the semi-discrete BKPESCS is given.

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1. Introduction

Soliton equations with self-consistent sources (SESCSs) have an important role in many fields of physics, such as hydrodynamics, plasma physics, solid-state physics [1–3], and they can describe the interactions between different solitary waves. For example, the KPESCS describes the interaction of a long wave with a short-wave packet propagating along the x, y plane at an angle to each other [4], and the nonlinear Schrödinger ESCS can describe the nonlinear interaction of an electrostatic high-frequency wave with the ion acoustic wave in plasma [1, 3]. Hence it has become an interesting problem to study how to propose and solve SESCSs. There exist several ways to study SESCSs, such as the inverse scattering transform, the Darboux transformation and Hirota's method (see [5–15]). Recently, a new source generation procedure [16] has been found to construct and solve SESCSs based on the bilinear method. (This new method was called source generalization procedure at first. While considering the fact that the procedure enables one to introduce sources in integrable equations, we renamed the procedure as source generation.) There are mainly three steps involved in the procedure:

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(1) to express N-soliton solutions of a soliton equation without sources in the form of determinant or pfaffian with some parameters, i.e., arbitrary constants $c_{i,j}$.
(2) to introduce the corresponding determinants or pfaffians with some new parameters, say, arbitrary functions of one independent variable, e.g. $c_{i,j}(t)$.
(3) to seek coupled bilinear equations whose solutions are just these new generalized determinants or pfaffians. The coupled system is the so-called SESCS.
The source generation procedure has been successfully applied to some continuous and discrete AKP-type equations[16–18].

On the other hand, it is known that a new hierarchy of soliton equations[19] which is called the BKP hierarchy was discovered by Date *et al* in 1981, and BKP-type equations have been deeply investigated [20–22]. These kind of equations have a characteristic, i.e., their bilinear forms have solutions which can only be written in a pfaffian form; meanwhile, as for continuous AKP-type equations, their bilinear forms have solutions of the determinantal form, including Wronskian-type and Grammian-type. Based on this difference, it is natural to try to research BKP equations with self-consistent sources, through the source generation procedure. Hence a (2+1)-dimensional SK ESCS

$$\begin{aligned} 9u_t + u_{xxxxx} + 15(u_x u_{xx} + u u_{xxx}) + 45u^2 u_x - 5\int_{-\infty}^{x} u_{yy}\mathrm{d}x \\ - 15\left(u u_y + u_x \int_{-\infty}^{x} u_y \mathrm{d}x + u_{xxy}\right) = \sum_{j=1}^{K} (\varphi_{j,xx}\psi_j - \varphi_j \psi_{j,xx}), \end{aligned} \tag{1}$$

$$\varphi_{j,y} = \varphi_{j,xxx} + 3u\varphi_{j,x}, \qquad j = 1, 2, \ldots, K \tag{2}$$

$$\psi_{j,y} = \psi_{j,xxx} + 3u\psi_{j,x}, \qquad j = 1, 2, \ldots, K, \tag{3}$$

was obtained in [16], and the solution of the pfaffian form to system (1)–(3) was also given. The (2+1)-dimensional SK ESCS is an integrable coupled generalization of the (2+1)-dimensional SK equation. If we make a suitable choice, the (2+1)-dimensional SK ESCS can be reduced to the (2+1)-dimensional SK equation, and its solution is also transformed into the pfaffian solution of the (2+1)-dimensional SK equation. Besides, if we set $u_y = 0$, $\varphi_{j,y} = \psi_{j,y} = 0$, the above system is reduced to the following (1+1)-dimensional SK ESCS:

$$9u_t + u_{xxxxx} + 15(u_x u_{xx} + u u_{xxx}) + 45u^2 u_x = \sum_{j=1}^{K} (\varphi_{j,xx}\psi_j - \varphi_j \psi_{j,xx}), \tag{4}$$

$$\varphi_{j,xxx} = -3u\varphi_{j,x}, \qquad j = 1, 2, \ldots, K \tag{5}$$

$$\psi_{j,xxx} = -3u\psi_{j,x}, \qquad j = 1, 2, \ldots, K. \tag{6}$$

What has been mentioned above is just a continuous case concerning BKP-type equations. However, there has been no literature on BKP-type ESCS in discrete cases, as far as we are concerned. So it would be interesting to consider whether it is feasible to study discrete or semi-discrete BKP-type ESCS. The purpose of this paper is to construct a semi-discrete BKP-type equation with self-consistent sources, applying the source generation procedure. The semi-discrete BKP-type equation is given by [23]

$$u_{n+1,y} - u_{n,y} = u_{n+1,xx} + u_{n,xx} + (u_{n+1,x} + u_{n,x} + 1)(u_{n+1,x} - u_{n,x}) - \tfrac{1}{2}(\mathrm{e}^{u_{n+2}-u_n} - \mathrm{e}^{u_{n+1}-u_{n-1}}), \tag{7}$$

where the subscripts x and y denote differentiation with respect to the variables x and y, respectively. Applying the dependent variable transformation

$$u_n = \ln \frac{\tau_{n+1}}{\tau_n},$$

equation (7) can be transformed into the bilinear equation

$$\left[\left(D_y - D_x^2 - D_x - \tfrac{1}{2}\right) e^{\frac{1}{2}D_n} + \tfrac{1}{2} e^{\frac{3}{2}D_n}\right]\tau_n \bullet \tau_n = 0, \tag{8}$$

where D is the Hirota bilinear operator [24]

$$D_x^m f(x) \bullet g(x) = \frac{\partial^m}{\partial s^m} f(x+s)g(x-s)|_{s=0}, \qquad m = 0, 1, 2, \ldots,$$

and

$$\exp(\delta D_n) f(n) \bullet g(n) = f(n+\delta)g(n-\delta).$$

This paper is organized as follows. In section 2, a pfaffian solution of the semi-discrete BKP-type equation is found. The semi-discrete BKPESCS is proposed through the source generation procedure, and its pfaffian solution is derived in section 3. Besides, a bilinear Bäcklund transformation for the semi-discrete BKPESCS is obtained in section 4. Finally, the conclusion and discussions are given in section 5.

2. A pfaffian solution of the bilinear equation (8)

In this part, we give a pfaffian-type solution of equation (8). The pfaffian-type solution has the form

$$\tau_n = \mathrm{pf}(1, 2, \ldots, 2N)_n = \mathrm{pf}(\bullet)_n, \tag{9}$$

where the pfaffian entries are defined by

$$\mathrm{pf}(i, j)_n = c_{ij} + \sum_{k=-\infty}^{-1} [\varphi_i(k+n+1)\varphi_j(k+n) - \varphi_i(k+n)\varphi_j(k+n+1)],$$
$$i, j = 1, 2, \ldots, 2N, \tag{10}$$

with $c_{ij} = -c_{ji}$ being a constant and $\varphi_i(m) = \varphi_i(m, x, y)$ satisfying the differential–difference formulae

$$\frac{\partial \varphi_i(m)}{\partial x} = \frac{1}{2}[\varphi_i(m+1) - \varphi_i(m-1)], \tag{11}$$

$$\frac{\partial \varphi_i(m)}{\partial y} = \frac{1}{4}[\varphi_i(m+2) - \varphi_i(m-2)] + \frac{1}{2}[\varphi_i(m+1) - \varphi_i(m-1)]. \tag{12}$$

Before we prove that τ_n satisfies equation (8), we introduce new pfaffian elements defined as follows:

$$\mathrm{pf}(d_m, i)_n = \varphi_i(m+n), \qquad \mathrm{pf}(d_m, d_l)_n = 0, \qquad m, l \in Z, 1 \leqslant i \leqslant 2N.$$

Then we get the following formulae according to [24]:

$$\tau_{n+1} = \tau_n + \mathrm{pf}(d_0, d_1, \bullet)_n, \ \tau_{n-1} = \tau_n - \mathrm{pf}(d_{-1}, d_0, \bullet)_n,$$

$$\tau_{n+2} = \tau_n + \mathrm{pf}(d_0, d_1, \bullet)_n + \mathrm{pf}(d_1, d_2, \bullet)_n,$$

$$\frac{\partial \tau_n}{\partial x} = \frac{1}{2}\mathrm{pf}(d_{-1}, d_1, \bullet)_n, \qquad \frac{\partial \tau_{n+1}}{\partial x} = \frac{1}{2}\mathrm{pf}(d_0, d_2, \bullet)_n,$$

$$\frac{\partial^2 \tau_n}{\partial x^2} = \frac{1}{4}[\mathrm{pf}(d_{-1}, d_2, \bullet)_n + \mathrm{pf}(d_0, d_1, \bullet)_n - \mathrm{pf}(d_{-2}, d_1, \bullet)_n - \mathrm{pf}(d_{-1}, d_0, \bullet)_n],$$

$$\frac{\partial \tau_n}{\partial y} = \frac{1}{4}[\mathrm{pf}(d_{-2}, d_1, \bullet)_n + \mathrm{pf}(d_{-1}, d_2, \bullet)_n - \mathrm{pf}(d_{-1}, d_0, \bullet)_n - \mathrm{pf}(d_0, d_1, \bullet)_n] + \frac{1}{2}\mathrm{pf}(d_{-1}, d_1, \bullet)_n.$$

Substituting the above results into equation (8) yields the pfaffian identity

$$\mathrm{pf}(d_{-1}, d_0, d_1, d_2, \bullet)_n \mathrm{pf}(\bullet)_n - \mathrm{pf}(d_0, d_1, \bullet)_n \mathrm{pf}(d_{-1}, d_2, \bullet)_n + \mathrm{pf}(d_0, d_2, \bullet)_n \mathrm{pf}(d_{-1}, d_1, \bullet)_n - \mathrm{pf}(d_1, d_2, \bullet)_n \mathrm{pf}(d_{-1}, d_0, \bullet)_n = 0.$$

The above pfaffian identity can be proved by applying the following pfaffian identity [24]:

$$(a_1, a_2, a_3, a_4, \bullet)(\bullet) - (a_1, a_2, \bullet)(a_3, a_4, \bullet) + (a_1, a_3, \bullet)(a_2, a_4, \bullet) - (a_1, a_4, \bullet)(a_2, a_3, \bullet) = 0.$$

Therefore τ_n in (9) is a pfaffian solution of equation (8). Here the function τ_n is only a formal solution, and for application of the $\mathrm{pf}(i, j)_n$ in (10), we can take

$$\varphi_i(m) = \left(\frac{1+p_i}{1-p_i}\right)^m \exp\left\{\left(\frac{1}{1-p_i} - \frac{1}{1+p_i}\right)x + \left(\frac{1}{(1-p_i)^2} - \frac{1}{(1+p_i)^2}\right)y\right\},$$

where each p_i is a constant satisfying $p_i > 1$, then the N-soliton solution of equation (8) can be obtained. Here, if we choose $N = 1, c_{12} = 1$, we get the one-soliton solution of equation (8):

$$\tau_n = 1 + \frac{p_1 - p_2}{p_1 + p_2}\left(\frac{1+p_1}{1-p_1}\right)^n \left(\frac{1+p_2}{1-p_2}\right)^n \exp\left\{\left(\frac{1}{1-p_1} - \frac{1}{1+p_1} + \frac{1}{1-p_2} - \frac{1}{1+p_2}\right)x + \left(\frac{1}{(1-p_1)^2} - \frac{1}{(1+p_1)^2} + \frac{1}{(1-p_2)^2} - \frac{1}{(1+p_2)^2}\right)y\right\}.$$

3. The semi-discrete BKP equation with self-consistent sources

In this part, we will apply the source generation procedure to the bilinear equation (8). We first change the function τ_n in (9) into the form

$$f_n = \mathrm{pf1}(1, 2, \ldots, 2N)_n = \mathrm{pf1}(\bullet)_n, \tag{13}$$

where the pfaffian entries are defined by

$$\mathrm{pf1}(i, j)_n = C_{ij} + \sum_{k=-\infty}^{-1}[\varphi_i(k+n+1)\varphi_j(k+n) - \varphi_i(k+n)\varphi_j(k+n+1)], \quad i, j = 1, 2, \ldots, 2N.$$

In the above expression, each function $\varphi_i(m)$ still satisfies relations (11)–(12), and $C_{ij} = -C_{ji}$ satisfy

$$C_{ij} = \begin{cases} C_i(y), & i < j \quad \text{and} \quad j = 2N+1-i, \quad 1 \leqslant i \leqslant K \leqslant N, \\ c_{ij}, & i < j \quad \text{and} \quad j \neq 2N+1-i, \end{cases}$$

where each $C_i(y)$ is an arbitrary function of the variable y. Then the function f_n will no longer satisfy equation (8). Following the source generation procedure, we introduce new functions expressed by

$$g_{i,n} = \sqrt{\dot{C}_i(y)}\,\mathrm{pf1}(d_0, 1, \ldots, \hat{i}, \ldots, 2N)_n, \qquad i = 1, 2, \ldots, K, \tag{14}$$

$$h_{i,n} = \sqrt{\dot{C}_i(y)}\,\mathrm{pf1}(d_0, 1, \ldots, \widehat{2N+1-i}, \ldots, 2N)_n, \qquad i = 1, 2, \ldots, K, \tag{15}$$

where the dot denotes the derivative of $C_i(y)$ with respect to the variable y, and the new pfaffian elements are defined by

$$\mathrm{pf1}(d_m, i)_n = \varphi_i(m+n), \qquad \mathrm{pf1}(d_m, d_l)_n = 0, m, l \in Z, \qquad 1 \leqslant i \leqslant 2N.$$

The functions f_n, $g_{j,n}$ and $h_{j,n}$ so-defined will be shown to satisfy the bilinear equations

$$\left[\left(D_y - D_x^2 - D_x - \frac{1}{2}\right) \mathrm{e}^{\frac{1}{2}D_n} + \frac{1}{2}\mathrm{e}^{\frac{3}{2}D_n}\right] f_n \bullet f_n = 2\sum_{i=1}^{K} \sinh\frac{D_n}{2} g_{i,n} \bullet h_{i,n}, \tag{16}$$

$$(D_x - \sinh D_n) g_{i,n} \bullet f_n = 0, \qquad i = 1, 2, \ldots, K, \tag{17}$$

$$(D_x - \sinh D_n) h_{i,n} \bullet f_n = 0, \qquad i = 1, 2, \ldots, K. \tag{18}$$

For simplicity of the proof, we set

$$k_{j,n} = \mathrm{pf1}(d_0, 1, \ldots, \hat{j}, \ldots, 2N)_n, \qquad 1 \leqslant j \leqslant 2N.$$

Then we have the formulae

$$f_{n+1} = f_n + \mathrm{pf1}(d_0, d_1, \bullet)_n, \qquad f_{n-1} = f_n - \mathrm{pf1}(d_{-1}, d_0, \bullet)_n, \tag{19}$$

$$\begin{aligned} f_{n,y} = {} & \sum_{i=1}^{K} \dot{C}_i(y)\mathrm{pf1}(1, \ldots, \hat{i}, \ldots, 2N \hat{+} 1 - i, \ldots, 2N)_n + \frac{1}{2}\mathrm{pf1}(d_{-1}, d_1, \bullet)_n \\ & + \frac{1}{4}[\mathrm{pf1}(d_{-2}, d_1, \bullet)_n + \mathrm{pf1}(d_{-1}, d_2, \bullet)_n - \mathrm{pf1}(d_{-1}, d_0, \bullet)_n - \mathrm{pf1}(d_0, d_1, \bullet)_n], \end{aligned} \tag{20}$$

$$\begin{aligned} f_{n+1,y} = {} & \sum_{i=1}^{K} \dot{C}_i(y)\mathrm{pf1}(1, \ldots, \hat{i}, \ldots, 2N \hat{+} 1 - i, \ldots, 2N)_n \\ & + \sum_{i=1}^{K} \dot{C}_i(y)\mathrm{pf1}(d_0, d_1, 1, \ldots, \hat{i}, \ldots, 2N \hat{+} 1 - i, \ldots, 2N)_n \\ & + \frac{1}{4}[\mathrm{pf1}(d_0, d_3, \bullet)_n + \mathrm{pf1}(d_{-1}, d_2, \bullet)_n - \mathrm{pf1}(d_1, d_2, \bullet)_n - \mathrm{pf1}(d_0, d_1, \bullet)_n] \\ & + \frac{1}{2}\mathrm{pf1}(d_0, d_2, \bullet)_n + \frac{1}{4}\mathrm{pf1}(d_0, d_1, d_{-1}, d_2, \bullet)_n, \end{aligned} \tag{21}$$

$$k_{j,n+1} = \mathrm{pf1}(d_1, 1, \ldots, \hat{j}, \ldots, 2N)_n, \qquad k_{j,n-1} = \mathrm{pf1}(d_{-1}, 1, \ldots, \hat{j}, \ldots, 2N)_n, \tag{22}$$

$$\frac{\partial k_{j,n}}{\partial x} = \frac{1}{2}[k_{j,n+1} - k_{j,n-1} - \mathrm{pf1}(d_{-1}, d_0, d_1, 1, \ldots, \hat{j}, \ldots, 2N)_n]. \tag{23}$$

Substitution of (19)–(22) into equation (16) comes to the sum of pfaffian identities:

$$\begin{aligned} \sum_{i=1}^{K} \dot{C}_i(y) & [\mathrm{pf1}(d_0, d_1, 1, \ldots, \hat{i}, \ldots, 2N \hat{+} 1 - i, \ldots, 2N)_n \mathrm{pf1}(\bullet)_n \\ & - \mathrm{pf1}(1, \ldots, \hat{i}, \ldots, 2N \hat{+} 1 - i, \ldots, 2N)_n \mathrm{pf1}(d_0, d_1, \bullet)_n \\ & - \mathrm{pf1}(d_1, 1, \ldots, \hat{i}, \ldots, 2N)_n \mathrm{pf1}(d_0, 1, \ldots, 2N \hat{+} 1 - i, \ldots, \bullet)_n \\ & + \mathrm{pf1}(d_0, 1, \ldots, \hat{i}, \ldots, 2N)_n \mathrm{pf1}(d_1, 1, \ldots, 2N \hat{+} 1 - i, \ldots, \bullet)_n] = 0, \end{aligned}$$

which indicates that f_n, $g_{j,n}$ and $h_{j,n}$ saitsfy equation (16). Similarly, by substituting (19) and (22)–(23) into equations (17) and (18), we get the following pfaffian identity: $(1 \leqslant j \leqslant 2N)$

$$\begin{aligned}\mathrm{pf1}(d_{-1}, d_0, d_1, 1, \ldots, \hat{j}, \ldots, 2N)_n \mathrm{pf1}(\bullet)_n - \mathrm{pf1}(d_{-1}, 1, \ldots, \hat{j}, \ldots, 2N)_n \mathrm{pf1}(d_0, d_1, \bullet)_n \\ + \mathrm{pf1}(d_0, 1, \ldots, \hat{j}, \ldots, 2N)_n \mathrm{pf1}(d_{-1}, d_1, \bullet)_n \\ - \mathrm{pf1}(d_1, 1, \ldots, \hat{j}, \ldots, 2N)_n \mathrm{pf1}(d_{-1}, d_0, \bullet)_n = 0.\end{aligned}$$

Therefore f_n, $g_{i,n}$ and $h_{i,n}$ in (13)–(15) are a kind of pfaffian solutions of equations (16)–(18). Also, equations (16)–(18) construct the bilinear form of equation (7) with self-consistent sources.

If we apply the dependent variable transformations:

$$u_n = \ln \frac{f_{n+1}}{f_n}, \qquad \phi_{i,n} = \frac{g_{i,n}}{f_n}, \qquad \psi_{i,n} = \frac{h_{i,n}}{f_n},$$

equations (16)–(18) are transformed into the nonlinear equations

$$\begin{aligned}u_{n+1,y} - u_{n,y} = u_{n+1,xx} + u_{n,xx} + (u_{n+1,x} + u_{n,x} + 1)(u_{n+1,x} - u_{n,x}) - \frac{1}{2}(\mathrm{e}^{u_{n+2}-u_n} - \mathrm{e}^{u_{n+1}-u_{n-1}}) \\ + \sum_{i=1}^{K} (\phi_{i,n+2}\psi_{i,n+1} - \phi_{i,n+1}\psi_{i,n+2} - \phi_{i,n+1}\psi_{i,n} + \phi_{i,n}\psi_{i,n+1}),\end{aligned} \tag{24}$$

$$\frac{\partial \phi_{i,n}}{\partial x} = \frac{1}{2}(\phi_{i,n+1} - \phi_{i,n-1})\,\mathrm{e}^{u_n - u_{n-1}}, \tag{25}$$

$$\frac{\partial \psi_{i,n}}{\partial x} = \frac{1}{2}(\psi_{i,n+1} - \psi_{i,n-1})\,\mathrm{e}^{u_n - u_{n-1}}. \tag{26}$$

4. Bilinear Bäcklund transformation for equations (16)–(18)

The semi-discrete BKPESCS (16)–(18) has a bilinear Bäcklund transformation described below.

Proposition 1. *The coupled system (16)–(18) has the bilinear Bäcklund transformation*

$$\left(\mathrm{e}^{\frac{1}{2}D_n} - \lambda\,\mathrm{e}^{-\frac{1}{2}D_n}\right) g_{j,n} \bullet f'_n = \left(\mu_j\,\mathrm{e}^{-\frac{1}{2}D_n} - \frac{\mu_j}{\lambda}\,\mathrm{e}^{\frac{1}{2}D_n}\right) f_n \bullet g'_{j,n}, \tag{27}$$

$$\left(\mathrm{e}^{\frac{1}{2}D_n} - \lambda\,\mathrm{e}^{-\frac{1}{2}D_n}\right) f_n \bullet h'_{j,n} = \left(\mu_j\,\mathrm{e}^{-\frac{1}{2}D_n} - \frac{\mu_j}{\lambda}\,\mathrm{e}^{\frac{1}{2}D_n}\right) h_{j,n} \bullet f'_n, \tag{28}$$

$$\left(D_x + \frac{\lambda}{2}\,\mathrm{e}^{-D_n} - \frac{1}{2\lambda}\,\mathrm{e}^{D_n} + \theta\right) f_n \bullet f'_n = 0, \tag{29}$$

$$\left(D_x + \frac{\lambda}{2}\,\mathrm{e}^{-D_n} - \frac{1}{2\lambda}\,\mathrm{e}^{D_n} + \theta\right) g_{j,n} \bullet g'_{j,n} = 0, \tag{30}$$

$$\left(D_x + \frac{\lambda}{2}\,\mathrm{e}^{-D_n} - \frac{1}{2\lambda}\,\mathrm{e}^{D_n} + \theta\right) h_{j,n} \bullet h'_{j,n} = 0, \tag{31}$$

$$\begin{aligned}\left(2D_y - 2D_x - \lambda D_x\,\mathrm{e}^{-D_n} - \frac{1}{\lambda} D_x\,\mathrm{e}^{D_n} - \lambda\theta\,\mathrm{e}^{-D_n} - \frac{\theta}{\lambda}\,\mathrm{e}^{D_n} - \nu\right) f_n \bullet f'_n \\ = 2\sum_{j=1}^{K}\left(\frac{\mu_j}{\lambda} g'_{j,n} h_{j,n} - \frac{\lambda}{\mu_j} g_{j,n} h'_{j,n}\right),\end{aligned} \tag{32}$$

here λ, θ, ν and μ_j are arbitrary constants.

Proof. Let f_n, $g_{j,n}$ and $h_{j,n}$ satisfy equations (16)–(18). What we need to prove is that f'_n, $g'_{j,n}$ and $h'_{j,n}$ satisfying (27)–(32) are solutions of equations (16)–(18). In fact, from relations (27)–(32) and the bilinear operator identities in appendix A, we have

$$
\begin{aligned}
P_1 &= \left[\left(\left(D_y - D_x^2 - D_x - \frac{1}{2}\right)e^{\frac{1}{2}D_n} + \frac{1}{2}e^{\frac{3}{2}D_n}\right]f_n\bullet f_n - 2\sum_{j=1}^{K}\sinh\frac{D_n}{2}g_{j,n}\bullet h_{j,n}\right](e^{\frac{1}{2}D_n}f'_n\bullet f'_n)\\
&\quad -(e^{\frac{1}{2}D_n}f_n\bullet f_n)\left[\left(\left(D_y - D_x^2 - D_x - \frac{1}{2}\right)e^{\frac{1}{2}D_n} + \frac{1}{2}e^{\frac{3}{2}D_n}\right]f'_n\bullet f'_n - 2\sum_{j=1}^{K}\sinh\frac{D_n}{2}g'_{j,n}\bullet h'_{j,n}\right]\\
&= 2\sinh\frac{D_n}{2}[(D_y - D_x)f_n\bullet f'_n]\bullet f_nf'_n - 2D_x\cosh\frac{D_n}{2}(D_xf_n\bullet f'_n)\bullet f_nf'_n\\
&\quad + \sinh\frac{D_n}{2}(e^{D_n}f_n\bullet f'_n)\bullet(e^{-D_n}f_n\bullet f'_n)\\
&\quad - 2\sum_{j=1}^{K}\left[\left(\sinh\frac{D_n}{2}g_{j,n}\bullet h_{j,n}\right)(e^{\frac{1}{2}D_n}f'_n\bullet f'_n) - (e^{\frac{1}{2}D_n}f_n\bullet f_n)\left(\sinh\frac{D_n}{2}g'_{j,n}\bullet h'_{j,n}\right)\right]\\
&= 2\sinh\frac{D_n}{2}[(D_y - D_x)f_n\bullet f'_n]\bullet f_nf'_n + \sinh\frac{D_n}{2}(e^{D_n}f_n\bullet f'_n)\bullet(e^{-D_n}f_n\bullet f'_n)\\
&\quad - \lambda\sinh\frac{D_n}{2}[(D_xf_n\bullet f'_n)\bullet(e^{-D_n}f_n\bullet f'_n) - f_nf'_n\bullet(D_x e^{-D_n}f_n\bullet f'_n)]\\
&\quad + \frac{1}{\lambda}\sinh\frac{D_n}{2}[(D_xf'_n\bullet f_n)\bullet(e^{-D_n}f'_n\bullet f_n) - f_nf'_n\bullet(D_x e^{-D_n}f'_n\bullet f_n)]\\
&\quad - 2\sum_{j=1}^{K}\left[\left(\sinh\frac{D_n}{2}g_{j,n}\bullet h_{j,n}\right)(e^{\frac{1}{2}D_n}f'_n\bullet f'_n) - (e^{\frac{1}{2}D_n}f_n\bullet f_n)\left(\sinh\frac{D_n}{2}g'_{j,n}\bullet h'_{j,n}\right)\right]\\
&= \sinh\frac{D_n}{2}\left[\left(2D_y - 2D_x - \lambda D_x e^{-D_n} - \frac{1}{\lambda}D_x e^{D_n} - \lambda\theta e^{-D_n} - \frac{\theta}{\lambda}e^{D_n}\right)f_n\bullet f'_n\right]\bullet f_nf'_n\\
&\quad + \sum_{j=1}^{K}\frac{\mu_j}{\lambda}[(e^{\frac{1}{2}D_n}f_n\bullet g'_{j,n})(e^{-\frac{1}{2}D_n}h_{j,n}\bullet f'_n - (e^{-\frac{1}{2}D_n}f_n\bullet g'_{j,n})(e^{\frac{1}{2}D_n}h_{j,n}\bullet f'_n)]\\
&\quad + \sum_{j=1}^{K}\frac{\lambda}{\mu_j}[(e^{\frac{1}{2}D_n}g_{j,n}\bullet f'_n)(e^{-\frac{1}{2}D_n}f_n\bullet h'_{j,n}) - (e^{-\frac{1}{2}D_n}g_{j,n}\bullet f'_n)(e^{\frac{1}{2}D_n}f_n\bullet h'_{j,n})]\\
&= \sinh\frac{D_n}{2}\left[\left(2D_y - 2D_x - \lambda D_x e^{-D_n} - \frac{1}{\lambda}D_x e^{D_n} - \lambda\theta e^{-D_n} - \frac{\theta}{\lambda}e^{D_n}\right)f_n\bullet f'_n\right.\\
&\quad \left. + 2\left(\frac{\lambda}{\mu_j}g_{j,n}h'_{j,n} - \frac{\mu_j}{\lambda}g'_{j,n}h_{j,n}\right)\right]\bullet f_nf'_n = \nu\sinh\frac{D_n}{2}f_nf'_n\bullet f_nf'_n = 0,\\
P_2 &= [(D_x - \sinh D_n)g_{i,n}\bullet f_n]g'_{j,n}f'_n - g_{j,n}f_n[(D_x - \sinh D_n)g'_{i,n}\bullet f'_n]\\
&= (D_xg_{j,n}\bullet g'_{j,n})f_nf'_n - g_{j,n}g'_{j,n}(D_xf_n\bullet f'_n) - \sinh\frac{D_n}{2}(e^{\frac{1}{2}D_n}g_{j,n}\bullet f'_n)\bullet(e^{-\frac{1}{2}D_n}f_n\bullet g'_{j,n})\\
&\quad + \sinh\frac{D_n}{2}(e^{\frac{1}{2}D_n}f_n\bullet g'_{j,n})\bullet(e^{-\frac{1}{2}D_n}g_{j,n}\bullet f'_n)\\
&= (D_xg_{j,n}\bullet g'_{j,n})f_nf'_n - g_{j,n}g'_{j,n}(D_xf_n\bullet f'_n)\\
&\quad - \sinh\frac{D_n}{2}\left[\lambda e^{-\frac{1}{2}D_n}g_{j,n}\bullet f'_n - \frac{\mu_j}{\lambda}e^{\frac{1}{2}D_n}f_n\bullet g'_{j,n}\right]\bullet(e^{-\frac{1}{2}D_n}f_n\bullet g'_{j,n})
\end{aligned}
$$

$$
\begin{aligned}
&+\sinh\frac{D_n}{2}\left(e^{\frac{1}{2}D_n}f_n\bullet g'_{j,n}\right)\bullet\left[\frac{1}{\lambda}e^{\frac{1}{2}D_n}g_{j,n}\bullet f'_n-\frac{\mu_j}{\lambda}e^{-\frac{1}{2}D_n}f_n\bullet g'_{j,n}\right]\\
&=(D_xg_{j,n}\bullet g'_{j,n})f_nf'_n-g_{j,n}g'_{j,n}(D_xf_n\bullet f'_n)\\
&\quad-\frac{\lambda}{2}[g_{j,n}g'_{j,n}(e^{-D_n}f_n\bullet f'_n)-(e^{-D_n}g_{j,n}\bullet g'_{j,n})f_nf'_n]\\
&\quad+\frac{1}{2\lambda}[g_{j,n}g'_{j,n}(e^{D_n}f_n\bullet f'_n)-(e^{D_n}g_{j,n}\bullet g'_{j,n})f_nf'_n]\\
&=\theta g_{j,n}g'_{j,n}f_nf'_n-\theta f_nf'_ng_{j,n}g'_{j,n}=0.
\end{aligned}
$$

The above results indicate that f'_n, $g'_{j,n}$ and $h'_{j,n}$ satisfy equations (16)–(17). Much in the same way, we can show that equation (18) holds for f'_n and $h'_{j,n}$. Hence we have completed the proof of the proposition. □

5. Conclusion and discussions

In this paper, we gave a pfaffian solution of the semi-discrete BKP-type equation (1) firstly. Then we have constructed and solved the semi-discrete BKP-type equation with self-consistent sources through the source generation procedure. When the coupled system has K pairs of sources, we can obtain its N-soliton ($N \geqslant K$) pfaffian solutions. If we set each $C_i(y)$ to be constant, the sources $g_{i,n}$ and $h_{i,n}$ in (14)–(15) become zero, and equations (16)–(18) are reduced to equation (8). Accordingly, f_n in (13) is reduced to the pfaffian solution of equation (2). Finally, we have given the bilinear Bäcklund transformation for the semi-discrete BKP-type equation with self-consistent sources, which can further explain the integrability of the coupled system. Here it should be noted that we have only given the soliton solutions of the semi-discrete BKP ESCS, on the other hand, some soliton equations and SESCSs, for example, the well-known KdV equation and KdV ESCS, have various solutions except for soliton type, such as complexiton, negaton and positon [25]–[27]. So we hope that more different kinds of solutions of this BKP-type ESCS can be derived, and this work is in progress. In addition, the new 'source generation procedure' can also be applied to the fully discrete BKP-type equations. This work will be reported in the future.

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Appendix. Hirota's bilinear operator identities

The following bilinear operator identities hold for arbitrary functions a, b, c, and d:

$$(D_xa\bullet b)cd-ab(D_xc\bullet d)=(D_xa\bullet c)bd-ac(D_xb\bullet d), \tag{A.1}$$

$$\sinh(\delta D_n)a\bullet a=0, \qquad D_t\cosh(\delta D_n)a\bullet a=0, \tag{A.2}$$

$$(e^{D_n}a\bullet b)cd-ab(e^{D_n}c\bullet d)=2\sinh\frac{D_n}{2}\left(e^{\frac{1}{2}D_n}a\bullet d\right)\bullet\left(e^{-\frac{1}{2}D_n}b\bullet c\right), \tag{A.3}$$

$$\left(D_ye^{\frac{1}{2}\delta D_x}a\bullet a\right)\left(e^{\frac{1}{2}\delta D_x}b\bullet b\right)-\left(e^{\frac{1}{2}\delta D_x}a\bullet a\right)\left(D_ye^{\frac{1}{2}\delta D_x}b\bullet b\right)=2\sinh\frac{\delta D_x}{2}(D_ya\bullet b)\bullet ab, \tag{A.4}$$

$$\left(D_y^2\,\mathrm{e}^{\frac{1}{2}D_n}a\bullet a\right)\left(\mathrm{e}^{\frac{1}{2}D_n}b\bullet b\right)-\left(\mathrm{e}^{\frac{1}{2}D_n}a\bullet a\right)\left(D_y^2\,\mathrm{e}^{\frac{1}{2}D_n}b\bullet b\right)=2D_y\cosh\frac{D_n}{2}(D_ya\bullet b)\bullet ab, \tag{A.5}$$

$$\left(\mathrm{e}^{\frac{3}{2}D_n}a\bullet a\right)\left(\mathrm{e}^{\frac{1}{2}D_n}b\bullet b\right)-\left(\mathrm{e}^{\frac{1}{2}D_n}a\bullet a\right)\left(\mathrm{e}^{\frac{3}{2}D_n}b\bullet b\right)=2\sinh\frac{D_n}{2}\left(\mathrm{e}^{D_n}a\bullet b\right)\bullet(\mathrm{e}^{-D_n}a\bullet b), \tag{A.6}$$

$$D_y\cosh\frac{D_n}{2}(\mathrm{e}^{-D_n}a\bullet b)\bullet ab=-\sinh\frac{D_n}{2}[(D_ya\bullet b)\bullet(\mathrm{e}^{-D_n}a\bullet b)-ab\bullet(D_y\,\mathrm{e}^{-D_n}a\bullet b)], \tag{A.7}$$

$$\begin{aligned}2D_x\cosh\frac{D_n}{2}(\mathrm{e}^{-D_n}a\bullet b)\bullet cd&=\mathrm{e}^{-\frac{1}{2}D_n}[(D_xa\bullet d)\bullet(\mathrm{e}^{-D_n}c\bullet b)-ad\bullet(D_x\,\mathrm{e}^{-D_n}c\bullet b)]\\&\quad+(D_x\,\mathrm{e}^{-D_n}a\bullet d)\bullet cb-(\mathrm{e}^{-D_n}a\bullet d)\bullet(D_xc\bullet b).\end{aligned} \tag{A.8}$$

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